PHYS 331 - Nov. 6, 2023

Last Time:

Fourier Transform:

$$\boxed{\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}$$

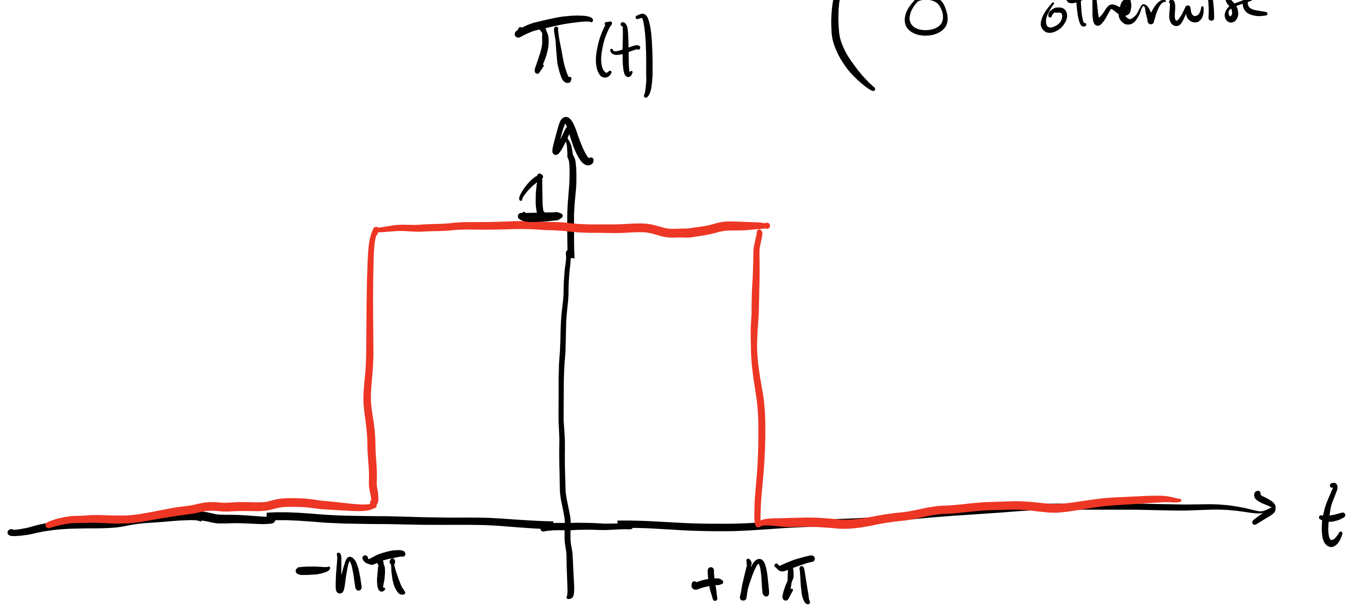
Converts fcn of time $f(t)$ to a fcn of frequency $\hat{f}(\omega)$.

Inverse Fourier Transform:

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{j\omega t} d\omega}$$

Converts fcn of freq. $\hat{f}(\omega)$ to a fcn of time $f(t)$

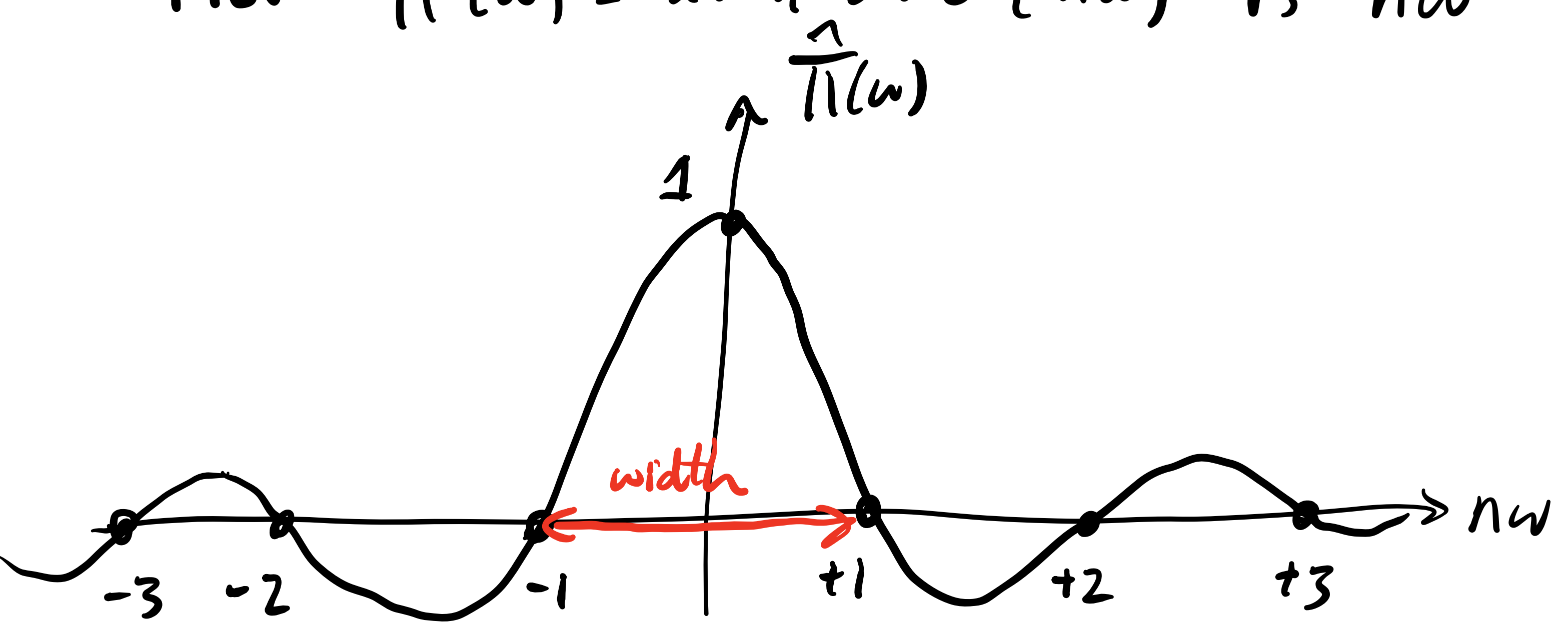
Eg. Box fun $\Pi(t) = \begin{cases} 1 & -n\pi < t < n\pi \\ \frac{1}{2} & t = \pm n\pi \\ 0 & \text{otherwise} \end{cases}$



$$\hat{\Pi}(\omega) = \int_{-\infty}^{\infty} \Pi(t)\, e^{-j\omega t}\, dt$$

$$= 2n\pi \operatorname{sinc}(n\omega) = 2n\pi \frac{\sin(n\pi\omega)}{n\pi\omega}$$

Plot $\hat{\Pi}(\omega) = 2n\pi \operatorname{sinc}(n\omega)$ vs $n\omega$



Sinc fcn is zero when $\sin(n\pi w) = 0$

$$\therefore\ n\cancel{\pi}w = \pm\cancel{\pi},\ \pm 2\cancel{\pi},\ \pm 3\cancel{\pi}, \ldots$$

$$nw = \pm 1, \pm 2, \ldots$$

$$w = \pm\frac{1}{n}, \pm\frac{2}{n}, \ldots$$

Characteristic width of sinc fcn deter. by the first two zero on either side of $nw = 0$.

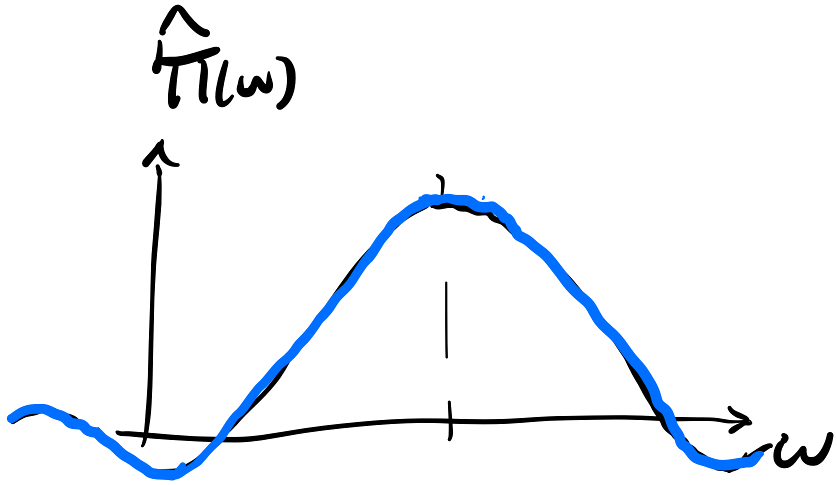
$$-1 < nw < +1$$

width, or freq span is $\underline{-\frac{1}{n} < w < \frac{1}{n}}$

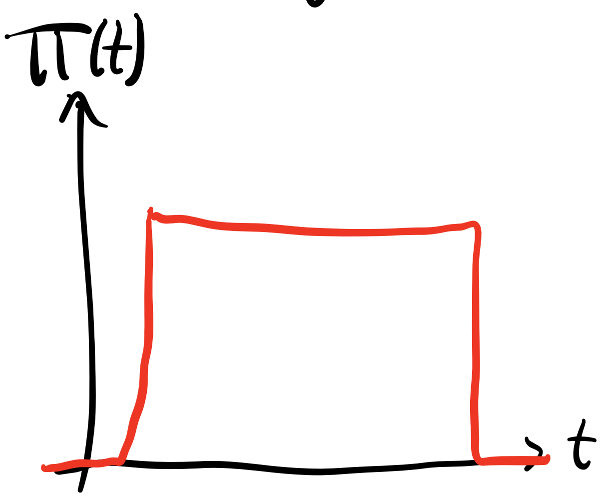
- large $n \Rightarrow$ wide fcn in time
  narrow fcn is w

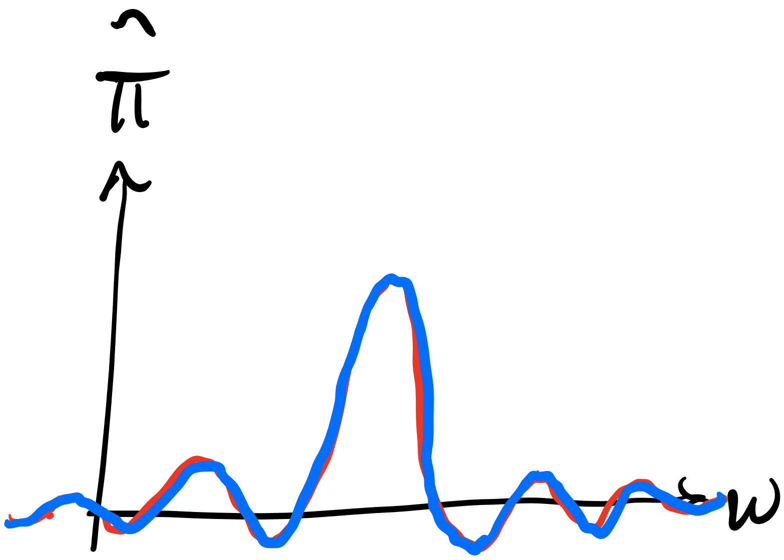
- small $n \Rightarrow$ narrow fcn of time $f(t)$
  wide fcn of w $\hat{f}(w)$

small n

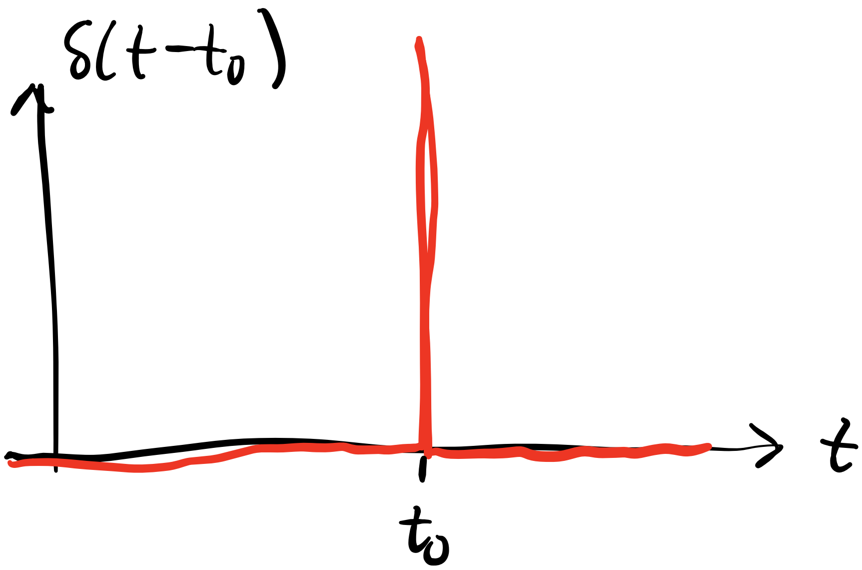


large n





Consider a delta fcn $\delta(t-t_0)$



$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt$$

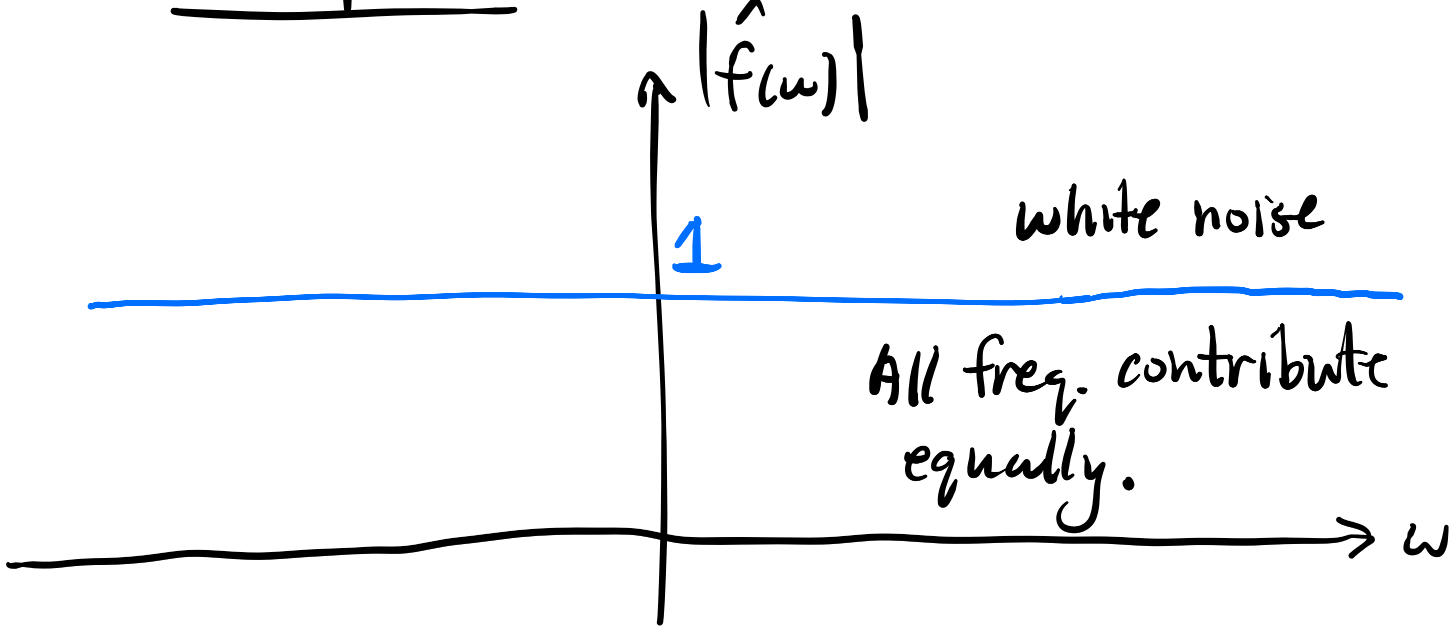
$$= e^{-j\omega t_0}$$

Recall

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, g(t)\, dt = g(t_0)$$

$$|\hat{f}(\omega)|^2 = \hat{f}(\omega)\, \hat{f}^*(\omega) = e^{-j\omega t_0}\, e^{j\omega t_0} = 1$$

$$|\hat{f}(\omega)| = 1$$



Consider the inverse fourier transform.

$$\underbrace{f(t)}_{\delta(t-t_0)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\hat{f}(w)}_{e^{-jwt_0}} e^{jwt}\, dw$$

$$\boxed{\therefore\ \delta(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jw(t-t_0)}\, dw}$$

One way of mathematically defining the delta fcn.

Properties of Fourier Transforms
(you will prove in Assign #3).

1. If $g(t) = f(t+b)$

then $\hat{g}(w) = e^{jwb}\hat{f}(w)$

where $\hat{f}(w)$ is Fourier Trans. of $f(t)$.

2. If $g(t) = f(at)$

then $\hat{g}(\omega) = \frac{1}{a} \hat{f}\left(\frac{\omega}{a}\right)$

3. $\underbrace{F[\dot{f}(t)]} = \int_{-\infty}^{\infty} \dot{f}(t) e^{-j\omega t} dt$

$= j\omega \hat{f}(\omega)$

Fourier trans. of $\frac{df(t)}{dt}$

One Additional Property of Fourier Transforms: Convolution Property.

Suppose $\hat{y}(\omega) = \hat{x}_1(\omega) \hat{x}_2(\omega)$

i.e. $\hat{y}$ can be expressed as a product of two fcns ω in which:

$$x_1(t) = F^{-1}[\hat{x}_1(\omega)]$$

$$x_2(t) = F^{-1}[\hat{x}_2(\omega)]$$

inverse Fourier transform.

What is the inverse Fourier transform of $\hat{y}(\omega)$?

$$y(t) = F^{-1}[\hat{y}(\omega)]$$

$$= F^{-1}[\hat{x}_1(\omega)\,\hat{x}_2(\omega)] = ?$$

Answer:

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

These two integrals are equiv. & related by a change of variables.

The two integrals define the convolution of $x_1(t)$ & $x_2(t)$.

The convolution has a special notation:

$$x_1(t) * x_2(t) = (x_1 * x_2)(t)$$

<u>not</u> mult. It is the convolution of $x_1(t)$ & $x_2(t)$

$$= \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Convolution Theorem is:

if $\hat{y}(\omega) = \hat{x}_1(\omega)\, \hat{x}_2(\omega)$, then

$$y(t) = x_1(t) * x_2(t)$$

Proof: Strategy is to start w/

$y(t) = x_1(t) * x_2(t)$, then take

Fourier trans. & show that

$\hat{y}(\omega) = \hat{x}_1(\omega)\,\hat{x}_2(\omega)$

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau$$

$$\therefore \hat{y}(\omega) = \int_{-\infty}^{\infty} y(t)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau \right] e^{-j\omega t}\,dt$$

Switch order of integration. Valid provided anything take out of dt integral does not have an dependence on time.

$$\hat{y}(w) = \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(t-\tau) e^{-jwt} \, dt \, d\tau$$

Make sub. $u = t - \tau$ in dt integral

$$du = dt$$

$$t = u + \tau$$

$$\hat{y}(w) = \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(u) \underbrace{e^{-jw(u+\tau)}}_{e^{-jwu} e^{-jw\tau}} \, du \, d\tau$$

$$\therefore \hat{y}(w) = \underbrace{\int_{-\infty}^{\infty} x_1(\tau) e^{-jw\tau} \, d\tau}_{\hat{x}_1(w)} \underbrace{\int_{-\infty}^{\infty} x_2(u) e^{-jwu} \, du}_{\hat{x}_2(w)}$$

$$\therefore \boxed{\hat{y}(\omega) = \hat{x}_1(\omega)\, \hat{x}_2(\omega)}$$ QED.

The inverse Fourier Trans. of the product $\hat{x}_1(\omega)\, \hat{x}_2(\omega)$ is the convolution of $x_1(t)$ & $x_2(t)$.

An exercise for the student.

Suppose $y(t) = x_1(t)\, x_2(t)$

What is $\hat{y}(\omega) = F[y(t)]$?

Ans: $\hat{y}(\omega) = \frac{1}{2\pi}\, \hat{x}_1(\omega) * \hat{x}_2(\omega)$